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Normal Vector Models of the Geomagnetic Secular Variation during Late Tertiary Time [and Discussion]

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Normal vector models of the geomagnetic secular variation during late Tertiary time

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During late Tertiary time, the secular variation of the geomagnetic field vector at a number of widely separated sites can be modelled as the sum of the field vector of a randomly sampled isotropic normal dipole moment and a non-dipole field vector that is a function of the assumed source geometry. Here, the non-dipole field vector distribution is calculated in the limit for an infinite number of radial dipole sources on the core surface with a possible latitudinal bias in geographic distribution and a normal moment distribution that is invariant with respect to geographic location. The model therefore consists of four degrees of freedom, and for the usual case of unit vector data the number reduces to three because the dipole variance the and non-dipole source moment variance can be specified only as their ratios to the mean dipole moment. The resultant non-dipole field vectors are non-isotropically normal with zero mean (if and only if the mean of the source moment distribution is everywhere zero). For the assumed source geometry, the dipole–non-dipole sum is normally distributed with mean and covariance as functions of latitude. For direct comparison with the available directional data, the normal distribution is integrated over all possible vector magnitudes to yield the associated unconditional (unit vector) distribution.

1. Introduction

In most statistical models of the distribution of the angular variance of the palaeomagnetic field vector as a function of time, the field vector is assumed to belong to a unit vector population whose variance in magnitude can be ignored. Owing to experimental difficulty and the mineralogical complexity of most rocks, estimates of the palaeomagnetic field direction generally do not include reliable estimates of the vector magnitude. In most cases, therefore, a description of the data in terms of their unit vector directions alone seems appropriate and sufficient. A major difficulty inherent in unit vector analysis is the fact that even though the true (three-dimensional) vector distribution may be conceptually simple in form, the distribution of the directions of its constituents may be analytically intractable. For example, rotational symmetry about the mean vector direction (provided that it exists) is usually regarded as a necessary, but not sufficient, condition for analytical simplicity of a unit vector distribution; yet it is possible to construct many simple vector sets whose unit vectors do not possess such symmetry.

The two directional distributions that have been analysed in most detail are those of Fisher (1953) and Bingham (1964, 1974), including its special cases, the Dimroth–Watson distributions. Both Fisher's and Bingham's distributions may be defined as conditional distributions of the three-dimensional normal distribution. Fisher's distribution is the subset of the isotropic normal conditional upon a given vector length (Downs 1966; Downs & Gould 1967), whereas Bingham's distribution is the subset of a non-isotropic normal with zero mean conditional upon a given vector length (Bingham 1964). Fisher's distribution has the additional remarkable property that it is also very similar to the unconditional angular distribution of the isotropic normal integrated over all vector lengths.

Fisher's distribution has played a central role in the statistical analysis of palaeomagnetic data. In addition to its applications for statistical inference, the distribution has often served as the basis for models to explain the observed changes in the field direction with time – or more precisely, with random sampling in time. Although the angular distribution of palaeomagnetic field directions is not Fisherian at all locations, it has been noted that when the directions are transformed to virtual geomagnetic poles (v.g.ps), the distribution of the poles is usually more nearly Fisherian than that of the field directions themselves. By definition, the v.g.p. defines the axis of a geocentric dipole moment that gives rise to the observed field direction. The field–v.g.p. relation is therefore defined by the following simple transformation: given a field vector $\boldsymbol{h}$ measured at geographic latitude λ_0 and longitude ϕ_0, the corresponding virtual dipole moment $\boldsymbol{m}$ is

$$\boldsymbol{m} = a^3 UT\boldsymbol{h}, \tag{1}$$

where a is the radius of the Earth, U is the orthogonal matrix

$$U = \begin{bmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin\lambda_0 & 0 & \cos\lambda_0 \\ 0 & 1 & 0 \\ -\cos\lambda_0 & 0 & \sin\lambda_0 \end{bmatrix} \tag{2}$$

and T is the non-orthogonal, diagonal matrix

$$T = \operatorname{diag}\,(1,\ -1,\ -\tfrac{1}{2}). \tag{3}$$

(The negative signs in T result from the conventional definition of the v.g.p. as the south magnetic pole.) Conversely, if the source of the field is a geocentric dipole, the field vector at all sites (λ_0, ϕ_0) is given by the inverse of (1):

$$\boldsymbol{h} = a^{-3}T^{-1}U^{-1}\boldsymbol{m}. \tag{4}$$

Equations 1 and 4 are used extensively in palaeomagnetic analyses to transform individual field directions to v.g.ps (or vice versa). In addition, they can be used to specify the transformation of the distribution of a set (or a conditional subset) of vectors from one coordinate system to the other. Consider, for example, a normal distribution of dipole moments with covariance A_m about the mean value $\boldsymbol{m}_0$:

$$p(\boldsymbol{m})\,\mathrm{d}v_m = \{(2\pi)^3 \det(A_m)\}^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(\boldsymbol{m}-\boldsymbol{m}_0)^{\mathrm{t}} A_m^{-1}(\boldsymbol{m}-\boldsymbol{m}_0)\}\,\mathrm{d}v_m, \tag{5}$$

where $\boldsymbol{m}^{\mathrm{t}}$ denotes the transpose (row) vector of the column vector $\boldsymbol{m}$. By (1),

$$(\boldsymbol{m}-\boldsymbol{m}_0) = a^3 UT(\boldsymbol{h}-\boldsymbol{h}_0), \tag{6}$$

where
$$\boldsymbol{h}_0 = a^{-3}T^{-1}U^{-1}\boldsymbol{m}_0, \tag{7}$$

so that
$$(\boldsymbol{m}-\boldsymbol{m}_0)^{\mathrm{t}} A_m^{-1}(\boldsymbol{m}-\boldsymbol{m}_0) = a^6(\boldsymbol{h}-\boldsymbol{h}_0)^{\mathrm{t}}\, TU^{-1}A_m^{-1}\,UT(\boldsymbol{h}-\boldsymbol{h}_0). \tag{8}$$

Therefore, a normal set of virtual dipole moments with mean $\boldsymbol{m}_0$ and covariance A_m maps into a normal set of field directions with mean $\boldsymbol{h}_0$ and covariance

$$A_h = a^{-6}T^{-1}U^{-1}A_m\,UT^{-1}. \tag{9}$$

For the special case of isotropic covariance $A_m = \sigma_0^2 I$, the distribution of the subset of $p(\boldsymbol{m})$ for all vectors of length r is precisely Fisherian with precision parameter $\kappa = r|\boldsymbol{m}_0|/\sigma_0^2$, and the distribution of all vectors, regardless of length, is approximately Fisherian with $\kappa \approx m_0^2/\sigma_0^2$. Although (9) ensures that the isotropic normal $p(\boldsymbol{m})$ maps into a normal $p(\boldsymbol{h})$, the operator T precludes the possibility that $p(\boldsymbol{h})$ is also isotropic (or that its directional distribution is Fisherian). The transformation of a surface of equal probability for the isotropically normal moments is illustrated sequentially in figure 1. For simplicity the mean dipole moment $\boldsymbol{m}_0$ is assumed to be parallel to the rotation axis, and the scaling constant a^{-6} is ignored. The shape

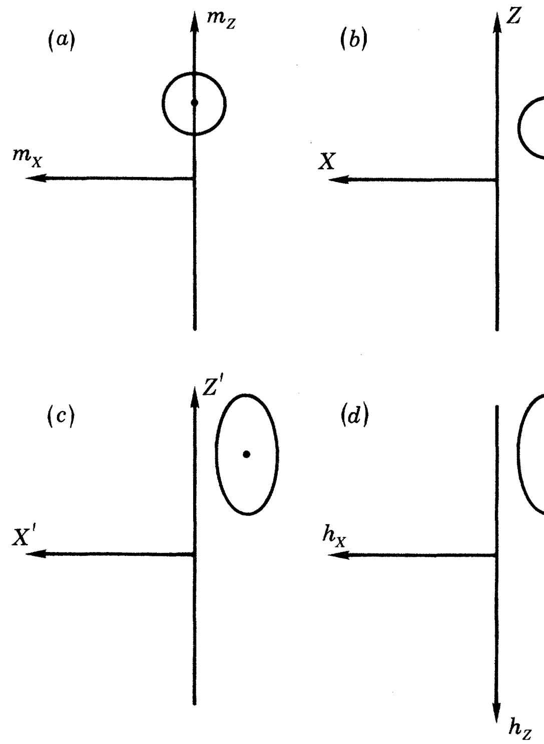

Figure 1. A sequential illustration of the transformation of an isotropic normal dipole moment population into the corresponding normal field vector population. ($a \rightarrow b$ represents the operator U^{-1}, and $b \rightarrow c \rightarrow d$ represents the operator T^{-1}.)

of the transformed surface is a geographically invariant prolate ellipsoid whose vertical semi-major axis is twice the horizontal semiminor axes. The orientation of the ellipsoid with respect to the mean vector $\boldsymbol{h}_0 = -m_0 a^{-3}(\cos\lambda_0, 0, 2\sin\lambda_0)$ does vary with geographic location, however, as does the configuration of the vector directions with respect to the mean direction. For $A_m = \sigma_0^2 I$, then, (9) is equivalent to model B proposed by Creer *et al.* (1959) and Creer (1962) to account for the geographic variation of the angular dispersion of the palaeomagnetic field vector.

2. The non-dipole contribution to secular variation

If A_m is isotropic in space and invariant in time, the field directions sampled at all locations, upon transformation to v.g.ps, would indicate a common and invariant Fisherian population. Whereas most v.g.p. data are more nearly Fisherian than the corresponding field directions, there is a significant increase of v.g.p. angular dispersion with increasing latitude (McElhinny

& Merrill 1975). The covariance of terms of higher degree than the geocentric dipole must therefore be included in the total covariance estimate. The generalization of model B to include the variances (and possible non-zero means) of harmonics g_i^j and h_i^j of degree $i > 1$ is impractical because of the number of degrees of freedom required, and because the appropriate moment to field transforms are unknown.

According to model D developed by Cox (1970) to account for the contribution of the non-dipole field, the total non-dipole variance is approximated as a uniform vector distribution (specifically, as a normal distribution with zero mean and isotropic covariance). Owing to the additive property of the normal distribution, it is then simple to determine the distribution of the vector sum of the dipole and non-dipole field vectors. Provided that the variance $\sigma_{\mathrm{nd}}^2 I$ of the non-dipole field is not a function of geographic location, only two degrees of freedom (σ_0^2/m_0^2, and $\sigma_{\mathrm{nd}}^2/m_0^2$ are required to describe the distribution of the unit vectors. In field space the vector mean is $\boldsymbol{h}_0 = -m_0 a^{-3}(\cos\lambda_0, 0, 2\sin\lambda_0)$ with covariance $A_h = \sigma_{\mathrm{nd}}^2 I + \sigma_0^2 a^{-6}$ diag $(1, 1, 4)$. Upon transformation to virtual dipole moment space, $\boldsymbol{m}_0 = m_0(0, 0, 1)$ with covariance

$$A_m = (\sigma_0^2 + \tfrac{1}{4}a^6\sigma_{\mathrm{nd}}^2)\, I + 3/4a^6\sigma_{\mathrm{nd}}^2 \begin{bmatrix} \sin^2\lambda_0 & 0 & -\frac{1}{2}\sin 2\lambda_0 \\ 0 & 1 & 0 \\ -\frac{1}{2}\sin 2\lambda_0 & 0 & \cos^2\lambda_0 \end{bmatrix}. \tag{10}$$

For $a^3\sigma_{\mathrm{nd}}, \sigma_0 \ll m_0$, it is easy to show that the angular variances of the unit vectors of the two distributions about their means are approximately

$$S_h^2 = a^{-6}(\sigma_0/m_0)^2 \left\{\frac{(5+3\sin^2\lambda_0)}{(1+3\sin^2\lambda_0)^2}\right\} + 2(\sigma_{\mathrm{nd}}/m_0)^2\,(1+3\sin^2\lambda_0)^{-1} \tag{11}$$

in field coordinates, and

$$S_m^2 = 2(\sigma_0/m_0)^2 + \tfrac{1}{4}a^6(\sigma_{\mathrm{nd}}/m_0)^2\,(5+3\sin^2\lambda_0) \tag{12}$$

in v.g.p. coordinates. The term $(1+3\sin^2\lambda_0)$ derives from the latitudinal variation of the mean field vector magnitude, whereas the term $(5+3\sin^2\lambda_0)$ results from the orientation of the distribution ellipsoids with respect to their respective mean vectors. Equations (11) and (12) were derived by Cox (1970), who stated that (12) is valid only if the field vectors are Fisherian. This is an unnecessary restriction since neither the field vectors nor the v.g.ps are in fact Fisherian. The validity of the derivation is based solely on the initial assumption that the total field vector consists of two components, a dipole that is isotropic about a non-zero mean (Fisherian) in moment space, and a non-dipole vector that is isotropic about a zero mean in field space. The ambiguity in Cox's derivation is then removed if his 'v.g.p. symmetric' operator transforms only the first term in (11) and his 'field symmetric' operator transforms only the second.

An important corollary of model D is that a sinusoidal oscillation of the dipole moment magnitude cannot be distinguished from a larger non-dipole variance by using only unit vector data. This corollary may be generalized to include any combination of sinusoids, so that in the limit a Fisherian unit vector is equivalent to a constant vector with isotropic white noise.

3. The non-dipole field as a function of its source

Model D suffers from two major deficiencies.

1. The predicted variation in the v.g.p. angular dispersion resulting from the non-dipole field alone is only 1 to 1.26 from the equator to the pole. This variation is much smaller than the variation observed in the present non-dipole field, and it is also smaller than the variation required to model the palaeomagnetic data. This deficiency may be circumvented by allowing σ_{nd} to increase with latitude, but that can be done only at the expense of ignoring the following objection.

2. Although it is mathematically expedient to model the field by the two simple distributions used in model D, it should be remembered that the choice of distributions is arbitrary and provides little information about the processes within the core by which the field is generated.

Here I shall attempt to modify models B and D by assuming that the variance of the field results from a stochastic assemblage of simple sources located in the outermost region of the core. For simplicity, the model is constructed so as to minimize the number of degrees of freedom. Collections of palaeomagnetic data from different latitudes may then be used to obtain the best estimates of the parameters necessary to specify the model. As in models B and D, it is assumed that the total moment, when averaged over a long interval of given polarity, is a geocentric axial dipole moment $\pm m_0(0, 0, 1)$. The variance about the corresponding mean field vector $\mp m_0 a^{-3}(\cos\lambda_0, 0, 2\sin\lambda_0)$ is assumed to consist of an intrinsic random normal perturbation of the dipole moment and an additional perturbation originating from transitory sources on the core surface.

As a first approximation, it is assumed that the core surface sources may be represented by radially oriented dipoles. Although the contribution of tangentially oriented dipoles has been investigated to some degree, it is difficult to characterize their distributions to fit the data with a small number of degrees of freedom. I shall therefore restrict the present discussion to the contribution of radial dipoles.

The field components observed at the Earth's surface at latitude and longitude (λ_0, ϕ_0) generated by a radial dipole of moment m_s, depth $(a-\alpha)$, and coordinates (λ_s, ϕ_s), $h_j = m_s a^{-3} f_j(\alpha/a, \lambda_s, \phi_s; \lambda_0, \phi_0)$ are given by Hurwitz (1960). For the core surface, $\alpha \approx \frac{1}{2}a$ so that we need to specify only the joint moment–geographic distribution $p_m(m_s, \lambda_s, \phi_s)$ of the dipoles in order to calculate the corresponding distribution of the field vectors $p_h(\boldsymbol{h}(\lambda_0, \phi_0))$ at all locations. At this point we must simplify p_s before proceeding quantitatively. The variance of the perturbing field must increase with latitude so that it is reasonable to assume either that the density of dipoles increases with latitude while their moment distribution is invariant or that their moment variance increases with latitude but their density is uniform on the core surface. Provided that the moment distribution is everywhere symmetric about zero (that is, inward and outward pointing dipoles are equally probable at all locations), the two simplifications are indistinguishable. Throughout this discussion I have assumed an invariant moment distribution, but the reader is cautioned that this simplification cannot be modified as readily as the alternative one to include the possibility of standing perturbations of the geocentric dipole. If m_s is independent of location, then $p_m(m_s, \lambda_s, \phi_s) = p_1(m_s)\, p_2(\lambda_s, \phi_s)$, and by choosing the simplest distributions that satisfy the criteria discussed above we have

$$p_1(m_s) = (2\pi\sigma_s^2)^{-\frac{1}{2}} \exp(-m_s^2/2\sigma_s^2) \tag{13}$$

and $$p_2(\lambda_s, \phi_s) = \kappa_s/(4\pi \sinh \kappa_s) \cosh (\kappa_s \sin \lambda_s), \tag{14}$$

where σ_s^2 is the moment variance and κ_s is the polar concentration parameter of the radial dipoles.

For discrete sources, the complexity of the Hurwitz transforms f_j precludes an analytical evaluation of $p_h(\boldsymbol{h})$, but if the number of sources is large, the distribution of $\boldsymbol{h}$ converges to the normal so that only the mean and covariance need to be calculated. Therefore, given that

$$\int_{-\infty}^{\infty} m_s p_1(m_s)\, dm_s = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} m_s^2 p_1(m_s)\, dm_s = \sigma_s^2,$$

then $$E\{h_j(\lambda_0, \phi_0)\} = 0 \quad \text{for all } \kappa_s \text{ and at all } (\lambda_0, \phi_0) \tag{15}$$

and

$$E\{h_i(\lambda_0, \phi_0) . h_j(\lambda_0, \phi_0)\} = a^{-6}\sigma_s^2 \int_0^{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p_2(\lambda_s, \phi_s) f_i(\tfrac{1}{2}, \lambda_s, \phi_s; \lambda_0, \phi_0) \times f_j(\tfrac{1}{2}, \lambda_s, \phi_s; \lambda_0, \phi_0) . \cos \lambda_s \, d\lambda_s \, d\phi_s, \tag{16}$$

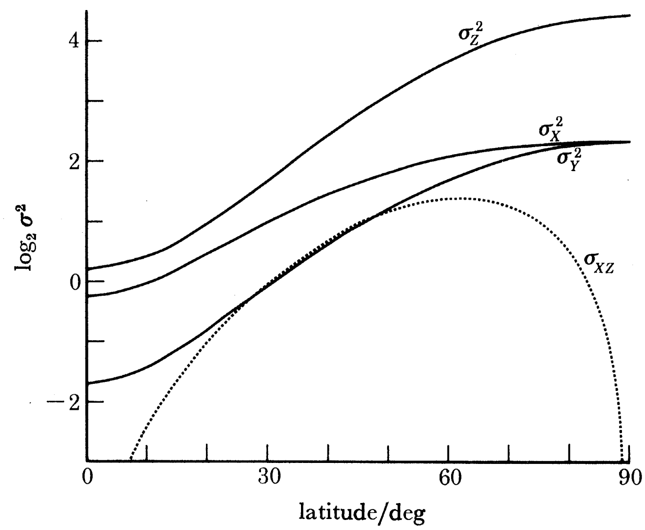

FIGURE 2. The variation of the non-zero covariance elements of $A(\lambda_0)$ as functions of latitude calculated for $\kappa_s = 5$ and $(\sigma_s/m_0)^2 = 0.028$.

so $\sigma_i^2 = E(h_i^2)$ and $\sigma_{ij} = E(h_i h_j)$, where $j = 1, 2, 3$ denotes the components x, y and z, respectively. Here we have neglected any electrical shielding by the core or lower mantle. Owing to the symmetry of $p_1(m_s)$, shielding has no effect on the calculation of $E(h_j)$, but it will alter the calculation of $E(h_i h_j)$ because additional weighting functions (and probably integration in additional variables) will be required.

For the uniform distribution of dipoles ($\kappa_s = 0$), the integrals of (16) may be evaluated analytically to yield $A_h = 1.918\sigma_s^2 a^{-6} \operatorname{diag}(1, 1, 2.943)$ at all sites (λ_0, ϕ_0). This result may be separated into the uniform and dipolar components

$$A_h = 0.676\, a^{-6}\sigma_s^2 \operatorname{diag}(1, 1, 1) + 1.242\, a^{-6}\sigma_s^2 \operatorname{diag}(1, 1, 4)$$

so that model D accommodates the trivial case of uniformly distributed dipoles provided that $a^6\sigma_{nd}^2/\sigma_0^2 \leqslant 0.544$. Note that when $a^6\sigma_{nd}^2/\sigma_0^2 = 0.544$ all the apparent dipole variance required by model D is provided by radial dipole variance.

For the longitudinally symmetric distribution $p_2(\lambda_s)$, the covariance elements σ_{xy} and σ_{yz} are identically zero for all κ_s and at all sites (λ_0, ϕ_0). The remaining four elements σ_x^2, σ_y^2, σ_z^2 and σ_{xz} are independent of longitude and may be calculated numerically or expressed analytically as a multiple summation in powers of κ_s and elementary functions of λ_0. The covariance elements as functions of latitude calculated with $\kappa_s = 5$ are shown in figure 2. The choice of $\kappa_s = 5$ as the optimum value results from the assumption of an intrinsic geocentric dipole wobble with angular standard deviation approximately 10° $(\sigma_0^2/m_0^2 \approx 0.015)$ in accordance with model D. The total covariance about $-m_0 a^{-3}(\cos\lambda_0, 0, 2\sin\lambda_0)$ is then

$$A_h = \sigma_0^2 a^{-6} \operatorname{diag}(1, 1, 4) + \sigma_s^2 a^{-6} A(\lambda_0), \tag{17}$$

where $A(\lambda_0)$ is the matrix of calculated covariances plotted in figure 2.

The four degrees of freedom of the model are κ_s, m_0, σ_0^2, and σ_s^2, and if no information regarding the magnitude of the field is available, only κ_s, $(\sigma_0/m_0)^2$ and $(\sigma_s/m_0)^2$ are required. Given $A(\lambda_0)$, the covariance in v.g.p. coordinates is easily calculated by using (9):

$$A_m = \sigma_0^2 I + \sigma_s^2 UTA(\lambda_0)\, TU^{-1}. \tag{18}$$

Because it is the distribution of the unit vectors (regardless of their individual magnitudes) that is to be compared with the data, the normal distribution (equation (5)) must be integrated over all $0 < m < \infty$ (or all h in field space) so that

$$p(\theta, \phi; \lambda_0) = \{(2\pi)^3 \det(A)\}^{-\frac{1}{2}} \int_0^\infty r^2 \mathrm{d}r \exp\{-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^{\mathrm{t}} A^{-1}(\boldsymbol{x}-\boldsymbol{x}_0)\}, \tag{19}$$

where

$r = |\boldsymbol{x}|$, $\boldsymbol{x} = \boldsymbol{m}$ (or $\boldsymbol{h}$), $A = A_m$ (or A_h), and $\boldsymbol{x}_0 = m_0(0, 0, 1)$ (or $-m_0 a^{-3}(\cos\lambda_0, 0, 2\sin\lambda_0)$)

in v.g.p. (field) coordinates. Although an analytical solution of this integral exists, its complexity renders it of little value (other than for computation), and it will not be discussed here.

The problem at hand is then to determine the set of parameters κ_s, $(\sigma_0/m_0)^2$ and $(\sigma_s/m_0)^2$ (provided that such a set exists) that optimizes the fit of $p(\theta, \phi; \lambda_0)$ to data collected at as many latitudes as possible. If an optimum set is determined, it is then simple to test each distribution of data against the theoretical distribution to determine the confidence with which the theoretical distribution can be rejected.

The density of (19) in colatitude θ was derived by Dodson (1980) with a Monte Carlo simulation in which 20 radial dipoles were used. Based upon data from Iceland (Watkins *et al.* 1977; Watkins & Walker 1977), the Canary Islands (Watkins 1973) and Hawaii (Doell & Cox 1965), the parameters $\kappa_s = 5$, $(\sigma_0/m_0)^2 = 0.015$ (10° 'dipole wobble') and $(\sigma_s/m_0)^2 = 0.028$ were chosen as optimum values. (For n identically distributed discrete dipoles σ_r^2, $\sigma_s^2 = n\sigma_r^2$; thus for 20 dipoles, $(\sigma_s/m_0)^2 = 0.028$ corresponds to $\sigma_r/m_0 = 0.0375$.) However, my initial conclusion, that the distributions of v.g.ps from higher latitudes more closely resemble the bipolar Dimroth–Watson distribution than they do Fisher's distribution, was merely coincidental. Despite their apparent similarity, the Dimroth–Watson distribution describes a conditional subset of normal vectors whose properties are very different from those of the unit vectors of (19), which is in fact much more closely related to Fisher's distribution.

Despite the goodness of fit of the proposed distribution to late Tertiary (approximately 13 Ma B.P. to present) data from the North Atlantic, the parameters specified above do not seem to fit as well the earlier data from the same region (e.g. the Miocene data from the Canary

Islands) nor data from widely separated localities, specifically the Hawaiian data. (For both data sets, the null hypothesis can be rejected with greater than 95% confidence.) The fit to the Hawaiian data is good (with less than 90% confidence for rejection), however, if the 'non-dipole' covariance $A(\lambda_0)$ at $\lambda_0 = 10°$ is used rather than the covariance predicted for Hawaii's true latitude ($\lambda_0 = 20°$) (while retaining the dipole co ariance predicted for the true latitude). Although the departure of the Hawaiian data from the predicted distribution may indicate a longitudinal asymmetry of $p_2(\lambda_s, \phi_s)$, as suggested by Doell & Cox (1972), an alternative explanation may be that one or more of the fitted parameters varies in time with wavelengths comparable with or longer than the record lengths. A crucial test is therefore whether variation in angular distribution is greatest for variation in longitude over a given time interval or for different time intervals at a given longitude.

4. Summary and concluding remarks

My original estimation of the parameters κ_s, $(\sigma_s/m_0)^2$ and $(\sigma_0/m_0)^2$ (Dodson 1980) was based upon an analysis of all the data (or all the data with intraflow consistency) provided by the references cited above. No data were excluded from the analysis on the basis of large angular distances from the population mean. Some of the low latitude poles are likely to have been recorded during transitions from one polarity to the other – an interval during which $(\boldsymbol{m}-\boldsymbol{m}_0)^2/\sigma_0^2 \gg 1$ and the constraints imposed upon the model are violated. It is therefore possible that the estimates of κ_s and $(\sigma_s/m_0)^2$ are somewhat larger than the values required to account for normal (non-transitional) secular variation.

Despite the rigid constraints imposed upon the model, the set of optimum parameters is not unique, or, more precisely, it cannot be considered unique within the limits of statistical resolution provided by currently available data. The choice of parameters $(\sigma_0/m_0)^2 = 0$ (no intrinsic geocentric dipole variance), $(\sigma_s/m_0)^2 = 0.038$ and $\kappa_s \approx 3.4$ yields a covariance $A(\lambda_0)$ that is very similar (but not identical) to the original A_h of (17) with $(\sigma_0/m_0)^2 = 0.015$, $(\sigma_s/m_0)^2 = 0.028$ and $\kappa_s = 5$. It is obvious, however, that there exists only a finite number of sets of optimum parameters. Since the observed angular variance of v.g.ps can nowhere be smaller than $2(\sigma_0/m_0)^2$, values for $(\sigma_0/m_0)^2$ much in excess of 0.028 are not permitted. Similarly, values for $\kappa_s < 3.4$ yield angular distributions whose variation with latitude is too small to account for the observed variation. These limitations place upper and lower permissible bounds on all three parameters.

The existence of a bounded space of permissible parameters has interesting implications. Consider the calculated covariance $A(\lambda_0)$ at the equator ($\lambda_0 = 0$) where $\sigma_{xz} = 0$ and σ_x^2, σ_y^2 and σ_z^2 are all minimum (for $\kappa_s > 0$). If $\sigma_{\min}^2 = \min(\sigma_x^2, \sigma_y^2, \frac{1}{4}\sigma_z^2)$ at $\lambda_0 = 0$, then we may write

$$A(\lambda_0) = A'(\lambda_0) + \sigma_{\min}^2 \operatorname{diag}(1, 1, 4) \tag{20}$$

such that $A'(\lambda_0)$ represents a purely non-dipole covariance and the second term is a purely dipole covariance. This apparent dipole variance is generated by the sources on the core surface independently of any intrinsic dipole variance. The total dipole variance therefore consists of two components, one of which is independent of the non-dipole field, and one of which is linearly dependent on the non-dipole field. The choice of $(\sigma_0/m_0)^2 = 0$, $(\sigma_s/m_0)^2 = 0.038$ and $\kappa_s = 3.4$ therefore results in complete coupling between the dipole and non-dipole variance, whereas the former choice $(\sigma_0/m_0)^2 = 0.015$, $(\sigma_s/m_0)^2 = 0.028$ and $\kappa_s = 5$ approximately minimizes the coupling between the two variances. Complete decoupling or negative coupling is not allowed by the model unless the variance of the core surface sources is dependent upon the intrinsic dipole variance.

The possibility of coupling between the dipole and non-dipole variances is relevant to the present discussion only if $(\sigma_s/m_0)^2$ or $(\sigma_0/m_0)^2$ varies with time. If, for example, σ_s varies with a wavelength comparable with the record length, then complete coupling will result in a greater variation in the distributions of data from different intervals (and common latitude) than will only partial coupling.

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Discussion

F. J. Lowes (*School of Physics, The University, Newcastle upon Tyne, U.K.*). Dr Dodson's random dipole model is essentially one of a white noise source at the core surface, the variance being a function of latitude. The estimation of the noise at the Earth's surface could be done by using a spherical harmonic analysis of the white noise source, and this would be easier if the latitudinal variation were that of a (sum of) Legendre polynomial(s) rather than that of a Fisher distribution.

R. E. Dodson. If the source functions $f_j(\alpha, \lambda_s, \phi_s; \lambda_0, \phi_0)$ and the source density function $p_2(\lambda_s, \phi_s)$ are expressed as sums of Legendre polynomials, then all the non-zero covariance coefficients at the surface can be calculated as multiple sums that include the integrals

$$\int_{-1}^{1} P_n^m(x)\, P_{n'}^m(x)\, P_{2k}^0(x)\, \mathrm{d}x.$$

Although these integrals are known for all n, n', k and m (the 'Gaunt integrals'), the summations are cumbersome and provide little gain in efficiency of calculation compared with direct numerical integration of the source functions. It should be pointed out, however, that in terms of the spherical harmonic expansions, the calculations are easily generalized to account for any distribution of sources, or for different source functions.